Isospin-Asymmetry Dependence
of the
Thermodynamic Nuclear Equation of State

Corbinian Wellenhofer
Technical University Munich

with: Jeremy W. Holt, Norbert Kaiser, Wolfram Weise

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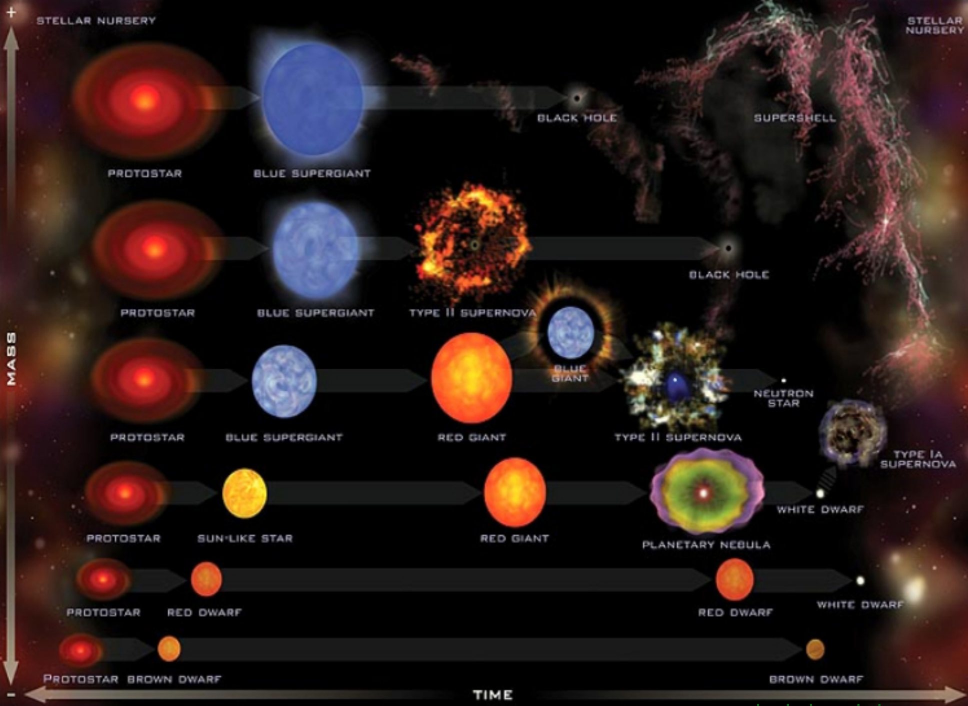

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RED GIANT
PLANETARY NEBULA
WHITE DWARF
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RED DWARF
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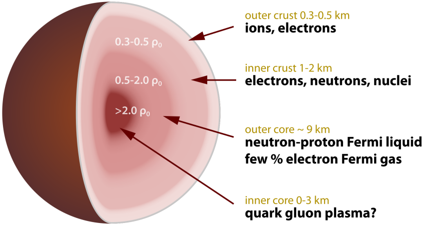
TIME

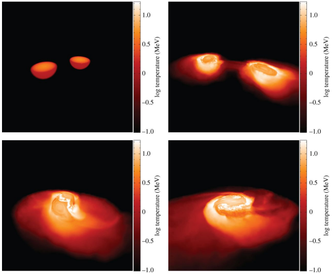
The Nuclear EoS: Interplay of Nuclear Physics and Astrophysics

Neutron Stars: $T \sim 0$



www2.astro.puc.cl

Binary Mergers: $T \lesssim 50$ MeV



Rosswog: Phil. Trans. R. Soc. A 371 (2013)

- astrophysical constraints on the EoS from (e.g.,) neutron-star masses and radii, moments of intertia, . . .
- task of nuclear theory: computation of the EoS from microphysics
  $\rightarrow$ EoS numerical input for simulations of supernovae and neutron-star mergers

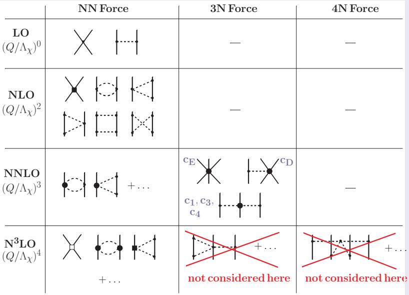
Novel developments in theoretical nuclear physics: chiral EFT, renormalization group

$\rightarrow$ low-momentum interactions (no "hard core")

$\rightarrow$ enables the use of Many-Body Perturbation Theory to compute the EoS

Modern Theory of Nuclear Interactions

- **chiral EFT:** general low-energy effective field theory consistent with symmetries of QCD, degrees of freedom: nucleons & pions
- **systematic hierarchy of nuclear interactions** controlled by expansion parameter Q/Λ_χ = soft scale/hard scale, where $\Lambda_\chi \sim 1$ GeV



- restrict resolution via UV cutoff $\Lambda < \Lambda_\chi$ in momentum space
- LECs $c_i(\Lambda)$ fixed by high-precision fits to few-nucleon observables

$\longrightarrow$ NN and 3N nuclear potentials for many-body calculations

Nuclear potentials are not unique! $\rightarrow$ uncertainty estimations (but: artifacts possible)

Low-momentum potentials $\Lambda \lesssim 450$ MeV: MBPT becomes valid approach!

Many-Body Perturbation Theory (MBPT)

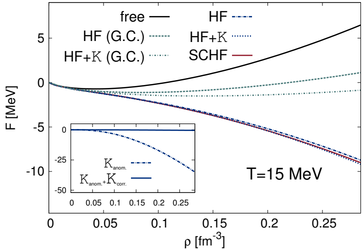
- linked-cluster expansion ("Goldstone expansion") for ground state energy ($T = 0$)
- textbook approach at finite T: expansion of grand-canonical potential
  $\Omega(T,\mu) = \Omega_0(T,\mu) + \Omega_1(T,\mu) + \Omega_2(T,\mu) + \mathcal{K}_{\mathrm{anom}}(T,\mu) + \ldots$
  But: not consistent with Goldstone expansion, cannot describe spinodal instability

Proper finite-temperature MBPT: canonical ensemble, expansion for free energy

- "naive" approach: linked-cluster expansion of free energy; does not work because canonical-ensemble averages are constrained (fixed N)
- instead: **evaluate ensemble averages via Legendre transform of cumulants**; gives
  $F(T,\tilde{\mu}) = F_0(T,\tilde{\mu}) + A_1(T,\tilde{\mu}) + A_2(T,\tilde{\mu}) + \mathcal{K}_{\mathrm{anom}}(T,\tilde{\mu}) + \mathcal{K}_{\mathrm{corr}}(T,\tilde{\mu}) + \ldots$
  - $\tilde{\mu}$ fixed by $\rho(T,\tilde{\mu}) = \partial F_0/\partial\tilde{\mu} \rightarrow$ consistent with Goldstone expansion!
  - additional contributions from **unlinked diagrams** $\mathcal{K}_{\mathrm{corr}}$, renormalize $\tilde{\mu}$
- canonical series can be also derived via reorganization of grand-canonical series (Kohn-Luttinger method), but equivalence is only formal (asymptotic series!)

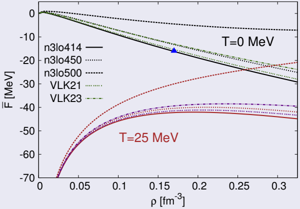
Mean-field benchmark

- fully renormalized MBPT: SCHF
- large $\mathcal{K}_{\mathrm{anom}}$ (ε renormalization), but $\mathcal{K}_{\mathrm{anom}} + \mathcal{K}_{\mathrm{corr}}$ (**ε and $\tilde{\mu}$** renormalization) is small
- spinodal region only from canonical and fully renormalized MBPT

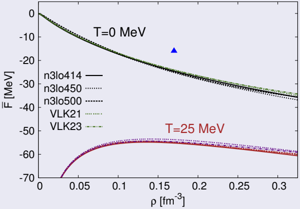


Chiral Nuclear EoS: Order-By-Order Results for various NN+3N Potentials

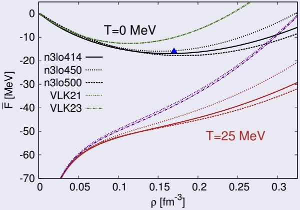
Isospin-symmetric nuclear matter: $\delta := (\rho_n - \rho_p)/\rho = 0$, $Y := \rho_p/\rho = 1/2$



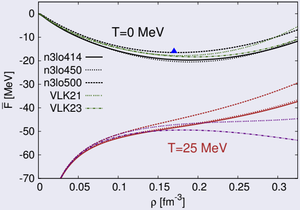
(a) NN first order, no 3N



(b) NN second order, no 3N



(c) NN second order, 3N first order

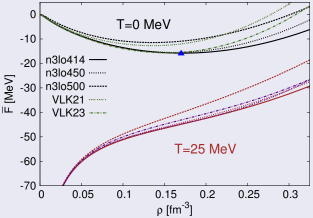


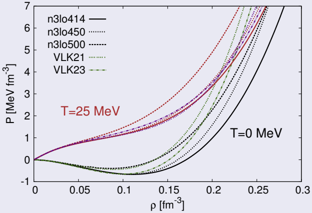
(d) NN second order, 3N second order

- n3lo414 & n3lo450: good perturbative behavior $F_1 > F_2 \gg F_3$ (third order: **Holt & Kaiser: 1612.04309 (2016)**)
- n3lo500: less perturbative ($F_{1,NN}$ & $F_{2,NN}$ similar magnitude)
- VLK21 & VLK23: NN perturbative, but large contributions from 3N potential

Chiral Nuclear EoS: Effective-Mass Improved Results

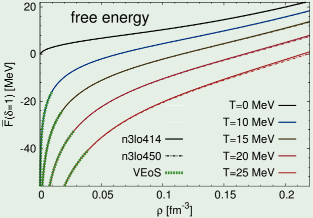
Isospin-symmetric nuclear matter: $\delta := (\rho_n - \rho_p)/\rho = 0$, $Y := \rho_p/\rho = 1/2$

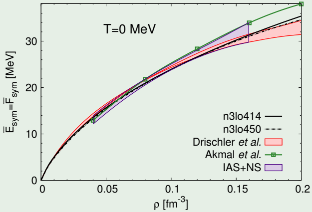




- empirical saturation point: n3lo414, n3lo450, ~~n3lo500~~, ~~VLK21~~, VLK23
- VLK21 & VLK23 ruled out by thermodynamics (pressure isotherm crossing)

Pure neutron matter ($\delta = 1$, $Y = 0$) — Symmetry free energy $\bar{F}_{sym}$



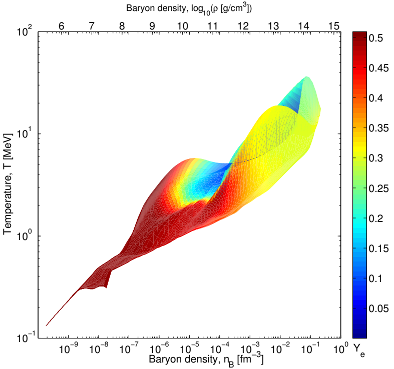


IAS&NS: Danielewicz & Lee, NPA 922 (2014)

Good agreement with virial expansion and constraints on $\bar{F}_{sym} := \bar{F}(\delta = 1) - \bar{F}(\delta = 0)$

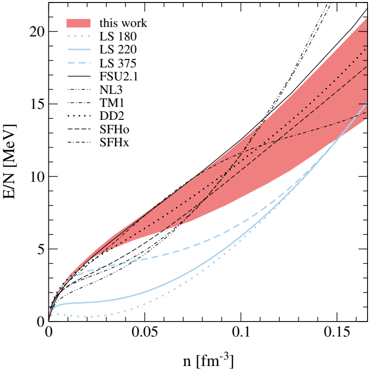
Need Chiral Nuclear EoS for Astrophysics Applications

Phase space covered in supernovae:
$T \sim 0 - 50$ MeV, $\rho \sim 0 - 6\,\rho_{\mathrm{sat}}$, $\delta \sim 0 - 1$



Fischer *et al.*: Astrophys. J. Suppl. 194 (2011)

Chiral vs. phenomenological EoS
($T = 0$, $\delta = 1$)



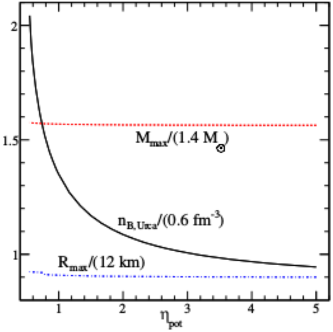
Krueger, Tews *et al.*: PRC 88 (2013)

Direct computation expensive; $\rightsquigarrow$ explicit parametrizations of the nuclear EoS?

$\rightsquigarrow$ parabolic approximation of δ dependence: $\bar{F}(\delta) \simeq \bar{F}(\delta = 0) + F_{\mathrm{sym}}\,\delta^2$

Question: is this really appropriate?

Isospin-Asymmetry Parametrizations beyond the Parabolic Approximation



Steiner: PRC 74 (2006)

Sensitivity to δ dependence of threshold density for direct URCA process:

- Quartic parametrization of EoS
  $F(\delta) = F(0) + A_2\delta^2 + A_4\delta^4$
- Change $A_{2,4}$ with $A_2 + A_4$ fixed
  $(\eta = 1/2, A_4 = -4/9A_2)$
  $(\eta = 1, A_4 = 0)$
  $(\eta = 3, A_4 = 4A_2)$

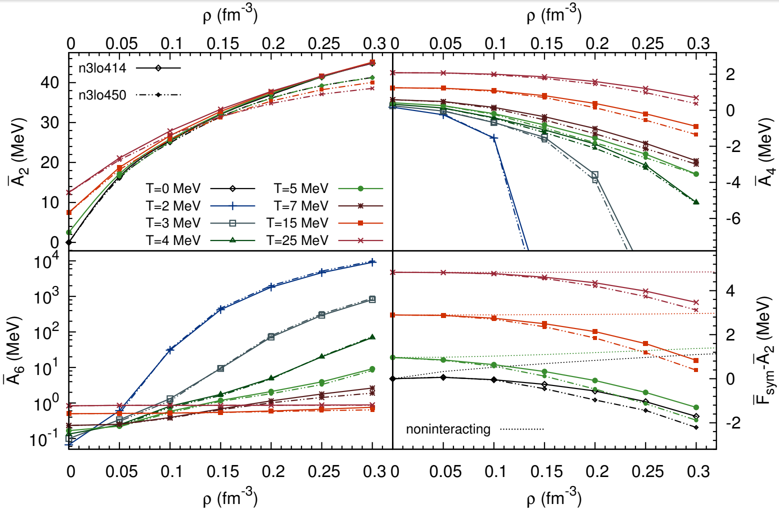
Terms beyond the parabolic approximation can be important for astrophysics!

$\rightsquigarrow$ investigate the expansion in δ:

$$F(T,\rho,\delta) \sim \sum_{n=0}^{N} A_{2n}(T,\rho)\,\delta^{2n}, \qquad \text{with} \qquad A_{2n}(T,\rho) = \frac{1}{(2n)!}\left.\frac{\partial^{2n}\bar{F}(T,\rho,\delta)}{\partial\delta^{2n}}\right|_{\delta=0}$$

- convergence behavior of the expansion?
- accuracy of parabolic approximation: how large is $F_{sym}(T,\rho) - A_2(T,\rho)$?

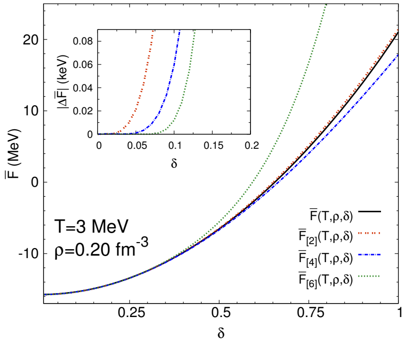
Results for Expansion Coefficients $A_{2,4,6}$ (and for $F_{sym} - A_2$)

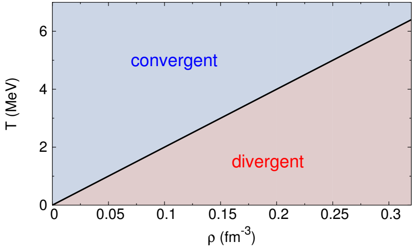


- **dominant contribution** to $F_{sym} - A_2$ from **noninteracting term and 3N interactions**
- $A_{2n\geq 4}$ become very large at low T $\rightsquigarrow$ divergent asymptotic expansion!
  $\Rightarrow$ higher-order parametrizations of δ dependence inhibited at low T

Divergent Expansion at Low Temperature

Examine higher-order approximations, e.g., $F_{[4]} := A_0 + A_2\delta^2 + A_4\delta^4$





What is the origin of the divergent behavior at low T?

→ contributions beyond Hartree-Fock in MBPT, e.g., second-order term:

$$F_2(T,\tilde{\mu}^{\mathrm{n}},\tilde{\mu}^{\mathrm{p}}) = -\frac{1}{8}\sum_{1234} \bar{V}_{2\mathrm{B}}^{12,34}\,\bar{V}_{2\mathrm{B}}^{34,12}\,\frac{n_1 n_2 \bar{n}_3 \bar{n}_4 - \bar{n}_1 \bar{n}_2 n_3 n_4}{\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2}$$

$T=0$: integrand diverges at boundary of integral, leads to $|A_{2n\geq 4}| \xrightarrow{T\to 0} \infty$

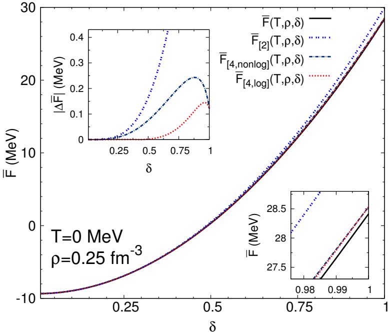
Exact results (at second order) with S-wave contact interaction

$$F(T=0,\rho,\delta) = A_0(0,\rho) + A_2(0,\rho)\,\delta^2 + \sum_{n=2}^{\infty} A_{2n,\text{reg}}(\rho)\,\delta^{2n} + \sum_{n=2}^{\infty} A_{2n,\log}(\rho)\,\delta^{2n}\ln|\delta|$$

- Logarithmic terms also from third-order terms **Holt & Kaiser: 1612.04309 (2016)**
- Logarithmic terms also when ladders are resummed (checked numerically)

Logarithmic terms also for chiral interactions? $\longrightarrow$ yes!

- $\bar{A}_{4,\log}$ and $\bar{A}_{4,\text{reg}}$ extracted numerically to good accuracy
- Quartic terms lead to considerably improved approximation!



Influence of higher-order terms (beyond δ^2) on neutron-star properties?

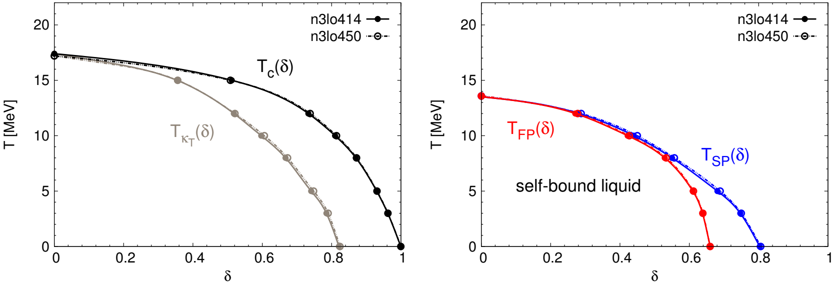
- logarithmic δ terms: only small influence on proton fraction
- influence of using Y parametrizations? (work in progress)

Isospin-Asymmetry Dependence of Nuclear Liquid-Gas Phase Transition

Stability criterion: $\mathcal{F}_{ij} = \frac{\partial^2 F(T,\rho_n,\rho_p)}{\partial\rho_i \partial\rho_j}$ has only positive eigenvalues

- $\delta = 0$: reduces to pure-substance criterion $\partial P/\partial\rho > 0$
- **isospin distillation** in isospin-**asymmetric** nuclear matter (binary system!)

- endpoint of **critical line** $T_c(\delta)$ at proton fraction $Y = (1-\delta)/2 \simeq 3 \cdot 10^{-4}$
- fragmentation temperature $T_{FP}(\delta)$ endpoint at $Y \simeq 0.17$



→ at large δ: $T_c(\delta)$ strongly influenced by entropy of mixing $\sim T\,Y\ln(Y)$
→ at $T = 0$: terms $\sim Y^{5/3}$ (also from interaction contributions!)

Summary and Outlook

Thermodynamic Nuclear EoS from Chiral EFT Interactions

- proper finite-temperature MBPT: **canonical** series, cumulants evaluated via Legendre transformation to **grand-canonical ensemble averages**
- accuracy of parabolic δ approximation decreased for high densities and high temperatures
- δ dependence is nonanalytic at low T, logarithmic terms at $T = 0$
- entropy of mixing $\sim TY \ln(Y)$, terms $Y^{5/3}$ at $T = 0$

Outlook: Chiral EoS for Astrophysics Applications

- **need to extrapolate EoS to higher densities and temperatures**
- one approach: construct explicit (ρ,T) parametrizations *via fits* ($\rightarrow$ quantify uncertainties of extrapolation via fit ambiguities)
- ρ dependence straightforward, but T dependence problematic:
  - extrapolation not well-behaved for many parametrizations (e.g., "Sommerfeld"-polynomial $\sum_n \alpha_n T^{2n}$)
  - SNM: computed data has tendency towards pressure isotherm crossing
  - PNM: approximately T-independent for $T \lesssim 25$ MeV and $\rho \lesssim 2\,\rho_{\text{sat}}$

Thank you for your attention!

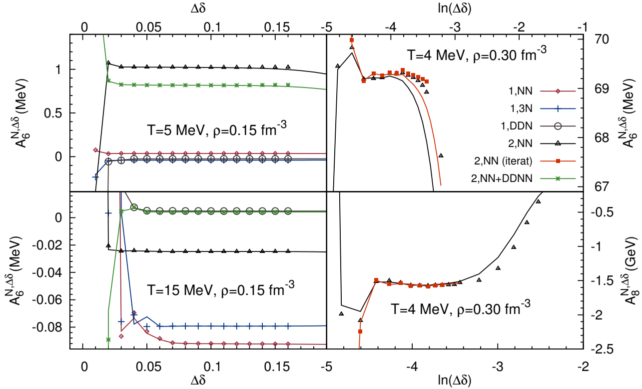
Appendix 1: Isospin-Asymmetry Expansions

A1: Extraction of Maclaurin Coefficients with Finite Differences

General of $2N+1$ point central finite difference approximation for $\bar{A}_{2n}(T,\rho)$

$$\bar{A}_{2n}(T,\rho) \simeq \bar{A}_{2n}^{N,\Delta\delta}(T,\rho) = \frac{1}{(2n)!\,(\Delta\delta)^{2n}} \sum_{k=0}^{N} \omega_{2n}^{N,k}\, \bar{F}(T,\rho,k\Delta\delta).$$

Fornberg: Math.Comp 51 (1988)



- stepsize ($\Delta\delta$) and grid length (N) variations as accuracy checks
- systematically increase precision of numerical integration routine

A2: Extraction of Leading Logarithmic Term at Zero Temperature

- finite differences of zero-temperature logarithmic series ($\sim \delta^{2n\geq 4} \ln|\delta|$):

$$\bar{A}_4^{N,\Delta\delta} = \bar{A}_{4,\text{reg}} + C_1^4(N)\bar{A}_{4,\log} + \bar{A}_{4,\log}\ln(\Delta\delta) + C_2^4(N)\bar{A}_{6,\log}\Delta\delta^2 + \mathcal{O}(\Delta\delta^4), \quad (0.1)$$

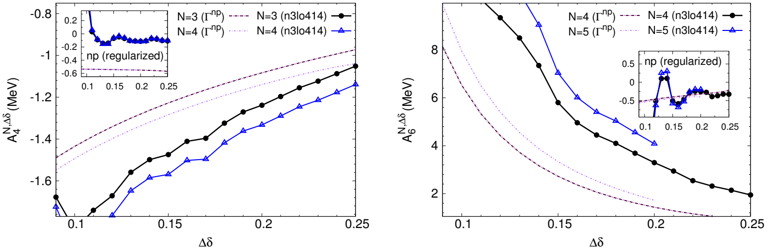
$$\bar{A}_6^{N,\Delta\delta} = \bar{A}_{6,\text{reg}} + C_1^6(N)\bar{A}_{4,\log}\Delta\delta^{-2} + \bar{A}_{6,\log}\ln(\Delta\delta) + C_2^6(N)\bar{A}_{6,\log} + \mathcal{O}(\Delta\delta^2). \quad (0.2)$$

- extract leading logarithmic term via:

$$\Xi_4(N_1, N_2) := \frac{\bar{A}_4^{N_1,\Delta\delta} - \bar{A}_4^{N_2,\Delta\delta}}{C_4^1(N_1) - C_4^1(N_2)} \simeq \bar{A}_{4,\log}, \quad (0.3)$$

$$\Xi_6(N_1, N_2) := \frac{\bar{A}_6^{N_1,\Delta\delta} - \bar{A}_6^{N_2,\Delta\delta}}{C_6^1(N_1) - C_6^1(N_2)}\Delta\delta^2 \simeq \bar{A}_{4,\log}, \quad (0.4)$$

- benchmark against analytical results for S-wave contact interaction



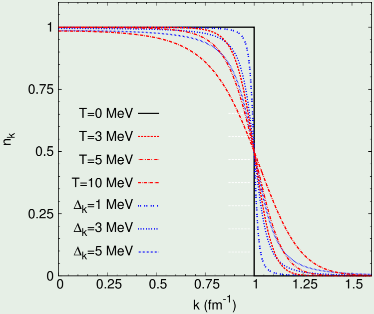
A3: BCS distribution functions

What about pairing? $\rightarrow$ perturbation series about BCS ground state ($\sim$ Bogoliubov)

BCS distribution functions:

$$n_k^{\mathrm{BCS}} = \frac{1}{2}\left[1 + \xi_k \left(\Delta_k^2 + \xi_k^2\right)^{-1/2}\right], \qquad \bar{n}_k^{\mathrm{BCS}} = \frac{1}{2}\left[1 - \xi_k \left(\Delta_k^2 + \xi_k^2\right)^{-1/2}\right]$$

$\rightarrow$ compare with finite-temperature Fermi-Dirac distributions



$\rightarrow$ expansion divergent also for BCS perturbation series, similar to low-T results

Overall:

- logarithmic terms for $T \rightarrow 0 \quad \wedge \quad \{N, \Omega\} \rightarrow \infty \quad \wedge \quad \Delta_k \rightarrow 0$
- divergent asymptotic expansion in the region "close enough" to these limits

Logarithmic terms also in self-consistent schemes, e.g., BHF, SCGF ?

$\rightarrow$ examine δ dependence of EoS from all-order-sum of ladder diagrams with S-wave contact interaction $V_{\text{contact}} = \pi M^{-1}(a_s + 3a_t + (a_t - a_s)\vec{\sigma}_1 \cdot \vec{\sigma}_2)$

Kaiser: EPJA 48 (2014)

$$\bar{E}_{0,\text{resum}}(k_F^n, k_F^p) = -\frac{24}{\pi M\left[(k_F^n)^3 + (k_F^p)^3\right]}\left(\Gamma^{nn}_{\text{resum}}(a_s) + \Gamma^{pp}_{\text{resum}}(a_s) + \Gamma^{np}_{\text{resum}}(a_s) + 3\Gamma^{np}_{\text{resum}}(a_t)\right)$$

where

$$\Gamma^{nn/pp}_{\text{resum}}(a_s) = \int_0^1 ds\, s^2 \int_0^{\sqrt{1-s^2}} d\kappa\, \kappa\, (k_F^{n/p})^5 \arctan \frac{I(s,\kappa)}{(a_s k_F^{n/p})^{-1} + \pi^{-1} R(s,\kappa)}$$

$$\Gamma^{np}_{\text{resum}}(a_{s/t}) = \int_0^{(k_F^n + k_F^p)/2} dP\, P^2 \int_{q_{\text{min}}}^{q_{\text{max}}} dq\, q \arctan \frac{\Phi(P, q, k_F^n, k_F^p)}{(a_{s/t})^{-1} + (2\pi)^{-1}\left[k_F^n R(\frac{P}{k_F^n}, \frac{q}{k_F^n}) + k_F^p R(\frac{P}{k_F^p}, \frac{q}{k_F^p})\right]}$$

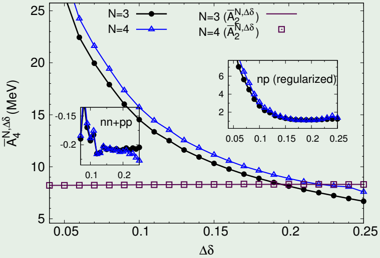
The functions $I(s,\kappa)$, $R(s,\kappa)$ and $\Phi(P, q, k_F^n, k_F^p)$ are given by

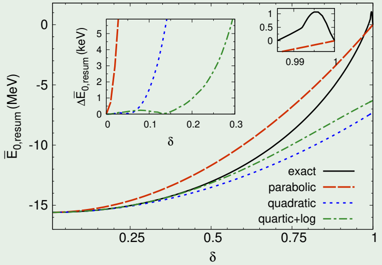
$$I(s,\kappa) = \kappa\, \Theta(1 - s - \kappa) + \frac{1 - s^2 - \kappa^2}{2s}\Theta(s + \kappa - 1),$$

$$R(s,\kappa) = 2 + \frac{1 - (s-\kappa)^2}{2s} \ln \frac{1 + s + \kappa}{|1 - s - \kappa|} + \frac{1 - s^2 - \kappa^2}{2s} \ln \frac{1 + s - \kappa}{1 - s + \kappa}$$

$$\Phi(P, q, k_F^n, k_F^p) = \begin{cases} q & \text{for} \quad P + q < k_F^p \\ \frac{(k_F^p)^2 - (P-q)^2}{4P} & \text{for} \quad k_F^p < P + q < k_F^n \ \wedge\ |P - q| < k_F^p \\ \frac{(k_F^n)^2 + (k_F^p)^2 - 2(P^2 - q^2)}{4P} & \text{for} \quad k_F^n < P + q \ \wedge\ P^2 + q^2 < \frac{(k_F^n)^2 + (k_F^p)^2}{2} \end{cases}$$

Ladder resummation: isospin-asymmetry dependence





Finite-difference results:

- quadratic coefficient $\bar{A}_2$ regular
- quartic coefficient $\bar{A}_4$: logarithmic for np-channel, but regular in nn+pp

Interaction contribution to ground-state energy per particle (np-channel):

- breakdown of "parabolic law", $\bar{F}_{\text{sym}} - \bar{A}_2 \simeq 15.6 - 8.3$ large
- quartic+log approximation very accurate for $\delta \lesssim 0.5$, but large deviation in very neutron-rich region
- exact results: maximum at $\delta \gtrsim 0.99$ (vanishes for larger a), and even a kink?

→ nonanalytic terms should arise also in self-consistent schemes, e.g., BHF, SCGF

Appendix 2: MBPT

Standard Finite-Temperature Perturbation Theory

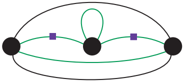
Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{V}$, with $\mathcal{H}\,|\Psi_p\rangle = E_p\,|\Psi_p\rangle$ and $\mathcal{T}\,|\Phi_p\rangle = \mathcal{E}_p\,|\Phi_p\rangle$, where

$\mathcal{T} = \sum_{ij} \langle\phi_i|T|\phi_j\rangle\, a_i^\dagger a_j = \sum_i \varepsilon_i a_i^\dagger a_i$ $\qquad$ $\mathcal{V} = \frac{1}{2!}\sum_{ijkl} \langle ij|V_{2N}|kl\rangle\; a_i^\dagger a_j^\dagger a_l a_k$

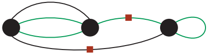
- grand-canonical partition function:
  $$Y = \sum_p \left\langle \Psi_p \,\middle|\, \mathrm{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})} \,\middle|\, \Psi_p \right\rangle = \sum_p \left\langle \Psi_p \,\middle|\, \mathrm{e}^{-\beta(\mathcal{T}-\mu\mathcal{N})}\,\mathscr{U}(\beta) \,\middle|\, \Psi_p \right\rangle$$
- Dyson operator:
  $$\mathscr{U}(\beta) = \mathrm{e}^{-\beta\mathcal{T}}\,\mathrm{e}^{\beta\mathcal{H}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\beta_n \cdots d\beta_1\; \mathcal{P}\left[\mathcal{V}_\mathrm{I}(\beta_n)\cdots\mathcal{V}_\mathrm{I}(\beta_1)\right]$$
- change basis ($\Psi_p \to \Phi_p$), then $\Delta A = A - \mathcal{A}$ given by:
  $$\Delta A = -\frac{1}{\beta} \ln\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\beta_n \cdots d\beta_1\; \left\langle \mathcal{P}\left[\mathcal{V}_\mathrm{I}(\beta_n)\cdots\mathcal{V}_\mathrm{I}(\beta_1)\right]\right\rangle\right]$$
- contraction rules (where $f_i^- = 1/\left[1+\exp\left(\beta(\varepsilon_i-\mu)\right)\right]$, $f_i^+ = 1 - f_i^-$):
  $$\underbrace{a_i^\dagger a_k} = \delta_{ik} f_i^- \quad \text{(hole)} \qquad\qquad \underbrace{a_k a_i^\dagger} = \delta_{jk} f_i^+ \quad \text{(particle)}$$
- linked-cluster theorem:
  $$\Delta A = \frac{1}{\beta}\sum_{n=0}^{\infty}(-1)^n \int_{\beta>\beta_n>\ldots>\beta_0>0} d\beta_n \cdots d\beta_0\; \left\langle \mathcal{V}_\mathrm{I}(\beta_n)\cdots\mathcal{V}_\mathrm{I}(\beta_0)\right\rangle_{\text{linked}}$$

Note: all this works also for canonical ensemble, but the constraint $\langle\Phi_p|\mathcal{N}|\Phi_p\rangle = N$ implies that single-particle states cannot be summed over independently $\Rightarrow$ useless!

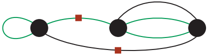
Third-Order Non-Skeletons



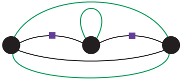
(a) self-energy



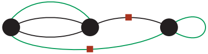
(b) one-loop



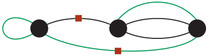
(c) one-loop



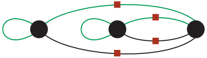
(d) self-energy



(e) one-loop

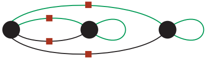


(f) one-loop

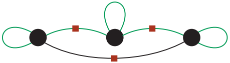


(a) two-loop

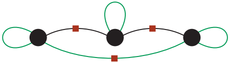
(b) two-loop



(c) two-loop



(d) three-loop



(e) three-loop

- non-skeletons are insertions: cut articulation lines $\rightsquigarrow$ collection of unlinked clusters
- "self-energy" and anomalous one-loop diagrams related by cyclic vertex permutations $\rightsquigarrow$ spurious terms in perturbation series

- $\mathcal{G}^{k_1\cdots k_m}_{i_1\cdots i_n} = \langle a^\dagger_{i_1} a_{i_1} \cdots a^\dagger_{i_n} a_{i_n} a_{k_1} a^\dagger_{k_1} \cdots a_{k_m} a^\dagger_{k_m} \rangle$ has generating function $\mathcal{Y}$:

$$\mathcal{G}^{k_1\cdots k_m}_{i_1\cdots i_n} = \frac{1}{\mathcal{Y}} \frac{\partial}{\partial(-\beta\varepsilon_{i_1})} \cdots \frac{\partial}{\partial(-\beta\varepsilon_{i_n})} \left(1 - \frac{\partial}{\partial(-\beta\varepsilon_{k_1})}\right) \cdots \left(1 - \frac{\partial}{\partial(-\beta\varepsilon_{k_m})}\right) \mathcal{Y}$$

- evaluate in terms of cumulants $\mathcal{K}_{i_1\ldots i_n} = \frac{\partial^n \ln \mathcal{Y}}{\partial(-\beta\varepsilon_{i_1})\cdots\partial(-\beta\varepsilon_{i_n})}$:

$$\mathcal{G}^{k_1\cdots k_m}_{i_1\cdots i_n} = \sum_{P\subset\{1,\ldots,m\}} (-1)^{|P|} \mathcal{G}_{i_1\cdots i_n \prod_{\nu\in P} k_\nu}, \qquad \mathcal{G}_{i_1\cdots i_n} = \sum_{\substack{P\in \text{ partitions} \\ \text{of } \{1,\ldots,n\}}} \prod_{I\in P} \mathcal{K}_{\prod_{\nu\in I} i_\nu}$$

- skeletons unchanged: $\mathcal{K}_{i_1\cdots i_n} = \mathcal{G}_{i_1\cdots i_n}$ for $i_1 \neq i_2 \neq \ldots \neq i_n$
- self-energy diagrams:

$$\mathcal{G}^{k_1\cdots k_m}_{i_1\cdots i_n aa} \sim \mathcal{K}_a\mathcal{K}_a + \mathcal{K}_{aa} = f_a^- f_a^- + f_a^- f_a^+ = f_a^-$$

$$\mathcal{G}^{k_1\cdots k_m}_{i_1\cdots i_n aaa} \sim \mathcal{K}_a\mathcal{K}_a\mathcal{K}_a + 3\mathcal{K}_{aa}\mathcal{K}_a + \mathcal{K}_{aaa} = f_a^- f_a^- f_a^- + 3 f_a^- f_a^+ + f_a^- f_a^+ (f_a^+ - f_a^-) = f_a^-$$

$$\vdots$$

- no contributions from anomalous diagrams:

$$\mathcal{G}^{a\cdots a}_{i_1\cdots i_n a\cdots a} = \sum_{P\subset\{1,\ldots,l\}} (-1)^{|P|} \mathcal{G}_{i_1\cdots i_n a\cdots a \prod_{\nu\in P} a_\nu} = \sum_{P\subset\{1,\ldots,l\}} (-1)^{|P|} \mathcal{G}_{i_1\cdots i_n a} = 0$$

Anomalous Contributions via Simply-Connected Diagrams

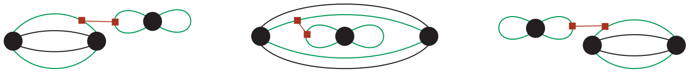
- expansion of logarithm yields

$$\Delta A = \sum_{n=0}^{\infty} \sum_{k} \sum_{\{a_i\},\{b_i\}} \beta^{b_1+\ldots+b_k-1} \binom{b_1+\ldots+b_k}{b_1,\ldots,b_k} \frac{(A_{a_1})^{b_1}\cdots(A_{a_k})^{b_k}}{b_1+\ldots+b_k}\Bigg|_{a_1 b_1+\ldots+a_k b_k=n}$$

- each term A_{a_i} has linked and unlinked contributions

$$A_{\mathrm{n,unlinked}} = \left[\frac{1}{\beta}\frac{1}{\alpha_1!\cdots\alpha_\nu!}(\Gamma_{n_1})^{\alpha_1}\cdots(\Gamma_{n_\nu})^{\alpha_\nu}\right]_{n_1^{\alpha_1}+\ldots+n_\nu^{\alpha_\nu}=n}$$

- by factorization theorem the only terms that survive are simply-connected in terms of higher cumulants, e.g.,
$(\Gamma_{2,\mathrm{normal}})^{kl}_{ij}(\Gamma_1)_{ab}\,:\;\left[\mathcal{G}^{kl}_{ij;ab}\right]_{\mathrm{s.c.}} = 2^2\times\delta_{ia}\mathcal{K}_{ii}\mathcal{K}_j\mathcal{K}_b\bar{\mathcal{K}}_l\bar{\mathcal{K}}_k - 2^2\times\delta_{ka}\mathcal{K}_{kk}\mathcal{K}_i\mathcal{K}_j\mathcal{K}_b\bar{\mathcal{K}}_l$



(a) $(\Gamma_{2,\mathrm{normal}}\Gamma_1)[\mathcal{G}^{kl}_{ij;ab}]_{\text{s.-c.}}$ **(b)** $(\Gamma_{2,\mathrm{normal}}\Gamma_1)[\mathcal{G}^{kl}_{ij;ab}]_{\text{s.-c.}}$ **(c)** $(\Gamma_{2,\mathrm{normal}}\Gamma_1)[\mathcal{G}^{kl}_{ij;ab}]_{\text{s.-c.}}$

resummation of insertions $\rightsquigarrow$ renormalization of distribution functions

$$f_a^{\pm}[\mathcal{S}] = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\mathcal{S}(a)\right)^m\frac{\partial^m}{\partial\varepsilon_a^m}f_a^{\pm} = \frac{1}{1+\exp\left(\pm\beta\left(\varepsilon_a+\mathcal{S}(a)-\mu\right)\right)}$$

Correlation-Bond Formalism

- start with standard perturbation series for $\Delta F = F - \mathcal{F}$:

$$\Delta F = \sum_{n=0}^{\infty} \sum_{k} \sum_{\{a_i\},\{b_i\}} \beta^{b_1+\ldots+b_k-1} \binom{b_1+\ldots+b_k}{b_1,\ldots,b_k} \frac{(F_{a_1})^{b_1}\cdots(F_{a_k})^{b_k}}{b_1+\ldots+b_k}\Bigg|_{a_1 b_1+\ldots+a_k b_k=n}$$

- the cumulants are now given by $\mathcal{K}_{i_1\ldots i_n} = \frac{\partial^n \ln \mathcal{Z}}{\partial[-\beta\varepsilon_{i_1}]\cdots\partial[-\beta\varepsilon_{i_n}]}$; evaluate using

$$\ln \mathcal{Z}(T,\tilde{\mu},\Omega) = \ln \mathcal{Y}(T,\tilde{\mu},\Omega) - \tilde{\mu}\frac{\partial \ln \mathcal{Y}(T,\tilde{\mu},\Omega)}{\partial\tilde{\mu}}$$

  where $\tilde{\mu}$ is given by $N(T,\tilde{\mu},\Omega) = \sum_i \tilde{f}_i^-$; since N is regarded fixed, $\tilde{\mu}$ is a functional of the spectrum $\{\varepsilon_\alpha\}$, i.e., as implicit equation:

$$\mathcal{J}(\tilde{\mu},\{\varepsilon_\alpha\}) = \sum_\alpha \tilde{f}_\alpha^- - N = 0$$

R. Brout & F. Englert, PR 120 (1960)

This method effectively "shifts" the constraint $\langle\Phi_p|\mathcal{N}|\Phi_p\rangle = N$ to the level of diagrams, resulting in new simply-connected contributions ("correlation bonds"), e.g., $\mathcal{K}_{ia} \sim \mathcal{K}_{ia} + \delta_{ia}\mathcal{K}_{ii}$, with

$$\mathcal{K}_{i_1 i_2} = \left(\frac{\partial\mathcal{K}_{i_1}}{\partial[-\beta\varepsilon_{i_2}]}\right)_{\mathcal{J}} = \frac{\partial\mathcal{K}_{i_1}}{\partial[\beta\tilde{\mu}]}\left(\frac{\partial[\beta\tilde{\mu}]}{\partial[-\beta\varepsilon_{i_2}]}\right)_{\mathcal{J}} = -\frac{\tilde{f}_{i_1}^-\tilde{f}_{i_1}^+\tilde{f}_{i_2}^-\tilde{f}_{i_2}^+}{\sum_\alpha \tilde{f}_\alpha^-\tilde{f}_\alpha^+}$$

Resummation of correlation bonds renormalizes auxiliary chemical potential $\tilde{\mu}$